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Bounds on the RMS Miss of Radar-Guided Missiles

Ilan Rusnak*
Rafael, Haifa 31021, Israel

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In preliminary analysis of guided systems, it is required to assess the miss distance performance from some small set of parameters. This paper presents analytical formulas of bounds on the achievable rms miss by a radar-guided missile against a variety of target maneuvers. These formulas use a set of core parameters that affect the miss distance; thus, they can be used for synthesis and analysis of the performance of radar-guided tactical missiles. The bound is derived, subject to the assumption that the missile guidance law and estimator are fully matched to the missile dynamics, the target maneuver, and the glint noise. The glint is the dominant noise source, the missile applies frequency agility, and there is no missile acceleration limit. No system can achieve smaller rms miss distance than the one presented, subject to the stated assumptions.

I. Introduction

THERE are many publications that give approaches and formulas to estimate the achievable miss distance [1–4]. No publication addresses the issue of assessment of the bound (greatest lower bound, GLB) that is achievable by a radar-guided missile with some minimal set of parameters (a set of core parameters). In this paper, such bounds on the achievable miss distance performance are derived.

By a bound, it is meant that formulas that describe the dependence of the achievable miss on a set of core parameters are given. By a set of core parameters, two types of parameters are addressed:

1) The first parameters are those that are not in the hands of the system engineer: the size, the acceleration level and velocity of the target, and the period of the encounter.

2) The second set of parameters are those that are in the hands of the system engineer: the sampling rate and the algorithms implementation (frequency agility implementation in the radar and guidance laws).

The assumptions in deriving the results are that 1) the scenario is linear, 2) the dominant noise is glint, 3) the radar seeker applies frequency agility, 4) the guidance law is matched to the missile's and the target's dynamics, 5) the missile has no acceleration constraint, and 6) the terminal phase period is sufficiently long, so that the initial conditions (heading error, etc.) have faded away.

II. Problem Statement and Approach to Solution

The most natural domain to pose the presented problem would be the domain of stochastic zero-sum differential games (noncooperative dynamic games) [5–8].

So formally, the solution value of (heuristically)

$$\min_{\text{missile's parameters}} \; \max_{\text{target's parameters}} J(\text{miss}) \tag{1}$$

is sought, where $J(\,)$ is the objective (say, the miss distance and the energy expenditure).

Specifically, it would be most appropriate to use the solution of the stochastic two-person zero-sum differential game with noisy measurements and system driving noise (winds gusts, pilot behavior, etc.). However, the existing status of the stochastic differential games theory does not provide a solution. Namely, although the problem is clearly stated, there is no known solution to this problem but only intermediate suboptimal solutions [9].

Moreover, today's targets do not perform evasive maneuvers based on the differential game theory; thus, although interesting, it does not have immediate practical value.

Following the previous discussion, in this work, it is assumed that the target performs a specific maneuver that is known to the missile. The missile applies the optimal guidance law and estimator based on one-sided optimization. The missile has no acceleration limit; that is, the linear optimal control problem is solved. This way, the GLB on the achievable miss is derivable.

The bound depends on a small set of parameters: the core parameters. It is our belief that the presented core parameters' set is the minimal set of parameters that give the bound on the achievable miss. Neither the minimality nor the uniqueness of this set is proved in the paper.

The glint is the dominant noise contributor to the miss from all the noises and disturbances (i.e., the effect of the thermal noise, the inertial sensors noise, etc. [1]) are not considered. This is further justified by the fact that the noises, other than glint (range dependent and independent noises), have zero contribution to the miss [3,10].

It is assumed that the missile implements frequency agility to decorrelate the glint noise, and the missile designer manages the sampling rate. In this case, the deterministic miss is negligible and, for all practical purposes, zero; thus, the miss is only due to the measurement noise: the glint.

III. Derivation of the Greatest Lower Bound

A. Linear Quadratic Gaussian Optimal Control Problem

A missile that implements guidance, control, and estimation based on the linear quadratic Gaussian optimal control problem (LQG) [8,11] is considered. The problem considered is a special case of this issue. The following is the presentation, for the completeness of the presentation in this paper, of the LQG problem and its solution.

We consider the linear stochastic system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + \Gamma w(t), \qquad x(t_o) = x_o \\ z(t) &= Cx(t) + v(t) \end{aligned} \tag{2}$$

where $x(t)$ is the state vector, $z(t)$ is the measurement, $u(t)$ is the input, $w(t)$ and $v(t)$ are the white Gaussian stochastic processes representing the system driving noise and the measurement noise, respectively, $x(t_o)$ is a Gaussian random vector, and

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*Research Fellow, (88), P.O. Box 2250; ilanru@rafael.co.il. AIAA Senior Member.

$$E[x_o] = \bar{x}_o, \qquad E[w(t)] = 0, \qquad E[v(t)] = 0$$
$$E[w(t)w(\tau)] = W\delta(t-\tau), \qquad E[v(t)v(\tau)] = V\delta(t-\tau)$$
$$E[w(t)v(\tau)] = 0, \qquad E[w(t)x_o] = 0, \qquad E[v(t)x_o] = 0$$
$$E\{[x_o - E(x_o)]^T[x_o - E(x_o)]\} = Q_o \tag{3}$$

All the vectors and matrices are of the appropriate dimensions.

The problem being considered here is of finding the optimal control $u^*(t)$ as a function of $(z(t), t_o \le t \le t_f\}$ that minimizes the quadratic criterion:

$$J = \frac{1}{2}E\left[x^T(t_f)Gx(t_f) + \int_{t_o}^{t_f}[z^T(t)Q_c z(t) + u^T(t)Ru(t)]\,\mathrm{d}t\right] \tag{4}$$

The solution is the cascade of a Kalman filter and a deterministic optimal controller [8,11]. This solution is based on the certainty equivalence principle and the separation theorem.

The Kalman filter is

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K(t)[z(t) - C\hat{x}(t)]$$
$$\hat{x}(t_o) = \bar{x}_o, \qquad K(t) = Q(t)CV^{-1}$$
$$\dot{Q}(t) = AQ(t) + Q(t)A^T + \Gamma W\Gamma^T - Q(t)C^TV^{-1}CQ(t)$$
$$Q(t_o) = Q_o \tag{5}$$

and the optimal controller (guidance law) is

$$u(t) = -F(t)\hat{x}(t)$$
$$F(t) = R^{-1}B^TP(t) - \dot{P} = P(t)A + A^TP(t)$$
$$+ C^TQ_cC - P(t)BR^{-1}B^TP(t)$$
$$P(t_f) = G \tag{6}$$

To derive the bound, the estimation error is considered [11]

$$e(t) = x(t) - \hat{x}(t) \tag{7}$$

and the dynamic equation of the combined system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} e(t) \\ \hat{x}(t) \end{bmatrix} = \begin{bmatrix} A - K(t)C & 0 \\ K(t)C & A - BF(t) \end{bmatrix}\begin{bmatrix} e(t) \\ \hat{x}(t) \end{bmatrix} + \begin{bmatrix} \Gamma & -K(t) \\ 0 & K(t) \end{bmatrix}\begin{bmatrix} w(t) \\ v(t) \end{bmatrix}\begin{bmatrix} e(t_o) \\ \hat{x}(t_o) \end{bmatrix} = \begin{bmatrix} x(t_o) - \hat{x}(t_o) \\ \bar{x}_o \end{bmatrix} = \begin{bmatrix} x_o \\ 0 \end{bmatrix} \tag{8}$$

We are interested in

$$E[x^T(t)Gx(t)] = \mathrm{trace}\{G[\bar{x}(t)\bar{x}^T(t) + Q_{\hat{x}\hat{x}}(t) + Q(t)]\}$$
$$\ge \mathrm{trace}\{G[Q_{\hat{x}\hat{x}} + Q(t)]\} \ge \mathrm{trace}[GQ(t)] \tag{9}$$

where

$$Q(t) = E[\{e(t) - E[e(t)]\}\{e(t) - E[e(t)]\}^T]$$
$$Q_{\hat{x}\hat{x}} = E[\{\hat{x}(t) - E[\hat{x}(t)]\}\{\hat{x}(t) - E[\hat{x}(t)]\}^T]$$
$$\bar{x}(t) = E[x(t)] \tag{10}$$

If symmetry in the initial condition is assumed (i.e., $E[x_o] = \bar{x}_o = 0$), then $\bar{x}(t) = E[x(t)] = 0$.

Further, from [8,11], the explicit solution is

$$Q_{\hat{x}\hat{x}}(t) = \int_{t_0}^{t}\Phi_{\hat{x}\hat{x}}(t,\tau)K(\tau)V(\tau)K^T(\tau)\Phi^T_{\hat{x}\hat{x}}(t,\tau)\,\mathrm{d}\tau \ge 0$$
$$\dot{\Phi}_{\hat{x}\hat{x}}(t,t_o) = [A - BF(t)]\Phi_{\hat{x}\hat{x}}(t,t_o), \qquad \Phi_{\hat{x}\hat{x}}(t_o,t_o) = I \tag{11}$$

As well, for Eq. (5), the explicit solution is [12]

$$Q(t) = \bar{Q} + \Phi(t,t_o)(Q_o - \bar{Q})$$
$$\times\left[I + \int_{t_0}^{t}\Phi^T(\tau,t_o)C^TV^{-1}C\Phi(\tau,t_o)(Q_o - \bar{Q})\,\mathrm{d}\tau\right]^{-1}\Phi^T(t,t_o)$$
$$\dot{\Phi}(t,t_o) = [A - \bar{Q}C^TV^{-1}C]\Phi(t,t_0),\ \Phi(t_o,t_o) = I$$
$$0 = A\bar{Q} + \bar{Q}A^T + \Gamma W\Gamma^T - \bar{Q}C^TV^{-1}C\bar{Q} \tag{12}$$

Now, for either 1) $Q_o \ge \bar{Q}$ or 2) the terminal phase period, $t_f - t_0$ is sufficiently long (i.e., the Kalman filter reaches steady state):

$$\lim_{t\to\infty}Q(t) = \bar{Q}$$

Assuming $E[x_o] = \bar{x}_o = 0$, we have, subsequently, from Eq. (9),

$$\overline{x^T(t)Gx(t)} = \mathrm{trace}\{G[Q_{\hat{x}\hat{x}} + Q(t)]\} \ge \mathrm{trace}[GQ(t)] \ge \mathrm{trace}[G\bar{Q}] \tag{13}$$

and the GLB we use here is

$$\overline{x^T(t)Gx(t)} = \mathrm{trace}[G\bar{Q}] \tag{14}$$

B. Optimal Guidance Law

The optimal deterministic control, the guidance law, is from the family of proportional navigation [3,8]. The deterministic performance of the modern guidance laws is intensively covered in the literature. The results from these deterministic analyses is that, in absence of guidance noises for sufficiently large acceleration constraint, the miss distance is very small, even for an accelerating target.

It is assumed that the measurements of the missile's states are noise free; thus, the guidance law is applied on the measured missile's states directly (no estimator is needed).

Thus, based on the previously mentioned observation experience, the dominant contributors to the miss are the guidance noises, which justify the use of the bound in Eq. (14).

C. Optimal Estimator

In a radar-guided missile, the most dominant noise source that is not under the control of the designer is the glint noise. The glint noise is a target-created noise, and it can be partially influenced by controlling the relative inertial angular rate of the line of sight (LOS), as expressed in the aircraft coordinates (by the designer of the guidance law) and by the radar designer by applying frequency agility [13] that decorrelates the glint noise measurements. Thus, the optimal estimator is a Kalman filter matched to the target maneuver and glint noise.

D. Greatest Lower Bound on Achievable Miss

From this discussion and Eq. (9), it follows that the miss is set by the performance of the Kalman filter. Specifically, if the covariance matrix Q, in Eq. (5), is the state estimation covariance, assuming that the state associated to the miss is the first state variable x_1, then the $Q_{11}(t_f)$ term is the covariance of the miss. Thus, the GLB on the rms miss is derived by assuming that $G = \mathrm{diagonal}([g \quad 0, \ \ldots, 0 \quad 0])$ and, from Eq. (14),

$$\mathrm{GLB\ (rms\ miss)} = \sqrt{Q_{11}(t_f)} \tag{15}$$

No other stochastic guidance law (or estimator), subject to the assumptions detailed previously, can achieve a smaller rms miss.

IV. Spectrum of Glint Noise

As stated previously, it is assumed that the dominant measurement noise is the glint noise. The standard deviation of the glint noise, for uniformly distributed reflectors, is [13]

$$\sigma_g^2 = \frac{1}{12}D^2, \qquad D = \sqrt{\frac{2}{\pi}}L \tag{16}$$

where D is the effective linear dimension of the target perpendicular to the target-missile LOS, and L is the linear dimension of the target perpendicular to the target-missile LOS.

When frequency agility is applied at rate of $f_s = 1/T_s$ Hz, the spectral density of (stair-type random stochastic process, independent and identically distributed sequence) the glint is given by [14]

$$V_{\rm go}(0) = \sigma_g^2 T_s\ \frac{\rm m^2}{\rm Hz}, \qquad V_{\rm go}(\omega) = V_{\rm go}(0)\left[\frac{\sin(\omega T_s/2)}{(\omega T_s/2)}\right]^2 \frac{\rm m^2}{\rm Hz} \tag{17}$$

where T_s is the sampling rate of the frequency agile radar.

Therefore, the spectral density of the measurement noise $v(t)$ is $V_{\rm go}$ $\rm m^2/Hz$.

V. Target-Maneuver Shaping Filter

In this work, the following target maneuvers from the large family of maneuvers considered in the literature are examined [3,15–19]: 1) the acceleration step maneuver (AS) [17,18], the 2) exponentially correlated acceleration (ECA) [17,18]–random-telegraph acceleration [19], 3) the jerk step maneuver (JS) [15,17,18], and 4) the random-telegraph jerk/exponentially correlated jerk (ECJ) [15].

VI. Target Acceleration Step Maneuver

For the target AS, the following dynamic model of the of the target-missile encounter [17,18] is

$$\frac{\rm d}{{\rm d}t}\begin{bmatrix} y(t) \\ \dot{y}(t) \\ a_T \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} y(t) \\ \dot{y}(t) \\ a_T \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} a_M(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w_T(t)$$

$$z(t) = [1 \quad 0 \quad 0]\begin{bmatrix} y(t) \\ \dot{y}(t) \\ a_T \end{bmatrix} + v(t) \tag{18}$$

where y is the target-missile separation distance, $\dot{y}$ is the target-missile separation velocity, a_T is the target acceleration, a_M is the missile acceleration, $w_T(t)$ is the target process driving noise, $v(t)$ is the target-missile separation measurement noise (glint), and z is the measured target-missile separation distance.

The preceding assumes that the target performs an evasive maneuver (a stochastic process) that is a step-acceleration maneuver of amplitude a_{T0}, for which the initiation instant is uniformly distributed in the interval $[t_o, t_f]$. The shaping filter [17,18] of this process is represented by

$$\frac{\rm d}{{\rm d}t} a_T = w_T(t) \tag{19}$$

where the spectral density of the target maneuver (the process noise) $w_T(t)$ is

$$W_{\rm AS} = \frac{a_{T0}^2}{t_m}\frac{(\rm m/s^3)^2}{\rm Hz} = \frac{\rm m^2}{\rm s^5} \tag{20}$$

where a_{T0} is the target step maneuver value, and $t_m = t_f - t_o$ is the time interval in which the target is expected to take an evasive maneuver.

The spectral density of the measurement noise $v(t)$ is $V_{\rm go}$ $\rm m^2/Hz$.

For this case, the algebraic Riccati equation (5) for the system in Eq. (18) has been solved explicitly [20]. The result is

$$Q = V_{\rm go}\begin{bmatrix} 2\omega_o & 2\omega_o^2 & \omega_o^3 \\ 2\omega_o^2 & 3\omega_o^3 & 2\omega_o^4 \\ \omega_o^3 & 2\omega_o^4 & 2\omega_o^5 \end{bmatrix}, \qquad \omega_o = \sqrt[6]{\frac{W_{\rm AS}}{V_{\rm go}}} \tag{21}$$

The GLB of the rms miss for the step-maneuvering target is from Eq. (15):

$$\text{rms miss m} \geq 0.5\ \sqrt[12]{\frac{T_s^5 D^{10} a_{T0}^2}{t_m}} \tag{22}$$

This is a practical GLB on the achievable rms miss. Further approximation gives

$$\text{rms miss m} \geq 0.5\sqrt{T_s}\ D\ \sqrt[12]{\frac{a_{T0}^2}{t_m}} \tag{23}$$

Remark: From the formula in Eq. (22), one might erroneously deduce that reducing the sampling period will reduce the miss distance. However, this is not true for the following reasons:

1) As the sampling period is reduced, the signal-to-noise ratio reduces as well, increasing the thermal noise, and thus violating the assumption of the glint noise dominance.

2) As the sampling period is reduced, the decorrelation due to application frequency agility will become invalid [13].

VII. Exponentially Correlated Acceleration Maneuver

For the random-telegraph target maneuver–ECA, the following dynamic model of the target-missile intercept–encounter is assumed. The ECA [17,18] has the same modeling as the random-telegraph target maneuver [19]; that is,

$$\frac{\rm d}{{\rm d}t}\begin{bmatrix} y(t) \\ \dot{y}(t) \\ a_T \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{\tau_T} \end{bmatrix}\begin{bmatrix} y(t) \\ \dot{y}(t) \\ a_T \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} a_M(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w_T(t)$$

$$z(t) = [1 \quad 0 \quad 0]\begin{bmatrix} y(t) \\ \dot{y}(t) \\ a_T \end{bmatrix} + v(t) \tag{24}$$

where τ_T is time constant of the target acceleration dynamics.

The preceding assumes that the target performs an evasive maneuver for which the acceleration is a random-telegraph stochastic process of amplitude a_{T0}. The shaping filter [17–19] of this process is represented by

$$\frac{\rm d}{{\rm d}t} a_T = -\frac{1}{\tau_T} a_T + w_T(t) \tag{25}$$

where the spectral density of the target maneuver (the process noise) $w_T(t)$ is

$$W_{\rm ECA} = \frac{2a_{T0}^2}{\tau_T}\frac{(\rm m/s^3)^2}{Hz} = \frac{\rm m^2}{\rm s^5} \tag{26}$$

For this case, the algebraic Riccati equation (5) for the system in Eq. (24) can be solved explicitly [20].

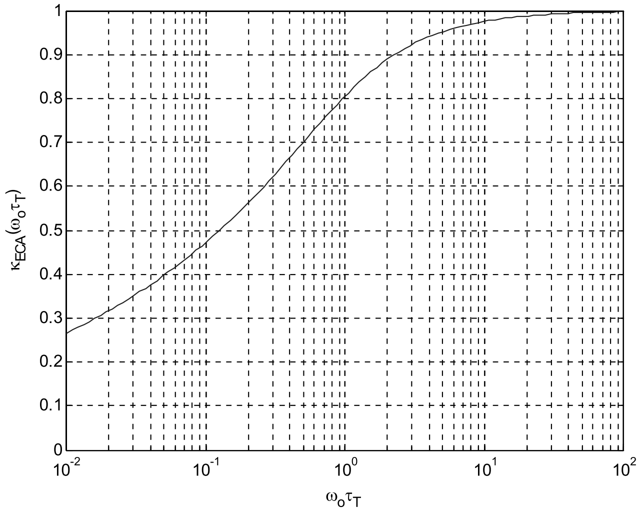

Fig. 1 The correction factor of the GLB miss distance computation for the ECA maneuver.

Then, from Eq. (15), by the use of the result from [20],

$$\bar{Q} = V_{\text{go}} \begin{bmatrix} 2\omega_o\kappa_{\text{ECA}}^2 & * & * \\ * & * & * \\ * & * & * \end{bmatrix}, \qquad \omega_o = \sqrt[6]{\frac{W_{\text{ECA}}}{V_{\text{go}}}}$$
$$\kappa_{\text{ECA}} = \kappa_{\text{ECA}}(\omega_o\tau_T) \tag{27}$$

From Appendix A,

$$\tau_T = 1.2\frac{V_T}{a_{T0}} \tag{28}$$

so that

$$\omega_o\tau_T = 1.98\,\sqrt[6]{\frac{V_T^5}{D^2T_s a_{T0}^3}} \tag{29}$$

The GLB of the rms miss for the ECA target maneuver is from Eq. (15):

$$\text{rms miss}[m] \geq 0.7\,\kappa_{\text{ECA}}(\omega_o\tau_T)\,\sqrt[12]{\frac{T_s^5 D^{10} a_{T0}^2}{\tau_T}}$$
$$= 0.69\,\kappa_{\text{ECA}}(\omega_o\tau_T)\,\sqrt[12]{\frac{T_s^5 D^{10} a_{T0}^3}{V_T}} \tag{30}$$

Figure 1 presents the correction factor $\kappa_{\text{ECA}}(\omega_o\tau_T)$ for the GLB miss distance computation for the ECA maneuver versus $\omega_o\tau_T$.

VIII. Target Jerk Step Maneuver

For the target JS (target ramp acceleration), the following model of the kinematics of the target-missile encounter [15] is assumed:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y(t) \\ \dot{y}(t) \\ a_T(t) \\ j_T(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} y(t) \\ \dot{y}(t) \\ a_T(t) \\ j_T(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} a_M(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w_T(t)$$
$$z(t) = [\,1 \quad 0 \quad 0 \quad 0\,]\begin{bmatrix} y(t) \\ \dot{y}(t) \\ a_T(t) \\ j_T(t) \end{bmatrix} + v(t) \tag{31}$$

where j_T is the target jerk $\mathrm{m/s^3}$.

The preceding assumes that the target performs an evasive maneuver (a stochastic process) that is a step jerk maneuver of amplitude j_{T0} for which the initiation instant is uniformly distributed in the interval $[t_o, t_f]$. The shaping filter [17,18] of this process is represented by

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}a_T = \frac{\mathrm{d}}{\mathrm{d}t}j_T = w_{\text{JS}}(t) \tag{32}$$

where the spectral density of the target maneuver (the process noise) $w_T(t)$ is

$$W_{\text{JS}} = \frac{j_{T0}^2}{t_m}\frac{(\mathrm{m/s^4})^2}{Hz} = \frac{\mathrm{m}^2}{\mathrm{s}^7} \tag{33}$$

where j_{T0} is the target jerk maneuver level.

For this case, the algebraic Riccati equation (5) for the system in Eq. (31) can be solved explicitly. The result is

$$\bar{Q} = V\begin{bmatrix} 2\sqrt{1+\frac{\sqrt{2}}{2}}\omega_o & 2\left(1+\frac{\sqrt{2}}{2}\right)\omega_o^2 & 2\sqrt{1+\frac{\sqrt{2}}{2}}\omega_o^3 & \omega_o^4 \\ 2\left(1+\frac{\sqrt{2}}{2}\right)\omega_o^2 & 2(1+\sqrt{2})\sqrt{1+\frac{\sqrt{2}}{2}}\omega_o^3 & (3+2\sqrt{2})\omega_o^4 & 2\sqrt{1+\frac{\sqrt{2}}{2}}\omega_o^5 \\ 2\sqrt{1+\frac{\sqrt{2}}{2}}\omega_o^3 & (3+2\sqrt{2})\omega_o^4 & 2(1+\sqrt{2})\sqrt{1+\frac{\sqrt{2}}{2}}\omega_o^5 & 2\left(1+\frac{\sqrt{2}}{2}\right)\omega_o^6 \\ \omega_o^4 & 2\sqrt{1+\frac{\sqrt{2}}{2}}\omega_o^5 & 2\left(1+\frac{\sqrt{2}}{2}\right)\omega_o^6 & 2\sqrt{1+\frac{\sqrt{2}}{2}}\omega_o^7 \end{bmatrix}, \qquad \omega_o = \sqrt[8]{\frac{W_{\text{JS}}}{V_{\text{go}}}} \tag{34}$$

Notice: The covariance matrices in Eqs. (21) and (34) are, as expected, symmetric ($a_{ij} = a_{ji},\ i,\ j = 1, 2, \ldots, n$). Moreover, the coefficients of the entries in the covariance matrices in Eqs. (21) and (34) are symmetric with respect to the other diagonal ($a_{ij} = a_{n+1-j,n+1-i}$). Such a matrix is called perisymmetric.

The GLB of the rms miss for the step jerk maneuvering target is from Eq. (15):

$$\text{rms miss}[m] \geq 0.54\,\sqrt[16]{\frac{T_s^7 D^{14} j_{T0}^2}{t_m}} \tag{35}$$

IX. Target Random-Telegraph Jerk

For the target random-telegraph jerk (ECJ), the following dynamic model of the target-missile intercept–encounter [15] is assumed. This is a modeling of the target ramp acceleration buildup with a time constant. That is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y(t) \\ \dot{y}(t) \\ a_T(t) \\ j_T(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{1}{\tau_T} \end{bmatrix}\begin{bmatrix} y(t) \\ \dot{y}(t) \\ a_T(t) \\ j_T(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} a_M(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w_T(t)$$

$$z(t) = [1 \quad 0 \quad 0 \quad 0]\begin{bmatrix} y(t) \\ \dot{y}(t) \\ a_T(t) \\ j_T(t) \end{bmatrix} + v(t) \tag{36}$$

The preceding assumes that the target performs an evasive maneuver for which the jerk is a random-telegraph stochastic process of amplitude j_{T0}. The shaping filter [17,18] of this process is represented by

$$\frac{\mathrm{d}}{\mathrm{d}t} j_T = -\frac{1}{\tau_T} j_T + w_{\mathrm{ECJ}}(t) \tag{37}$$

The time constant is τ_T and has the same value as for the ECA target-maneuver model in Eq. (28). From Appendix B,

$$j_{T0} = a_{T0}\omega_T = \frac{a_{T0}^2}{V_T} \tag{38}$$

where the spectral density of the target maneuver (the process noise) $w_T(t)$ is

$$W_{\mathrm{ECJ}} = \frac{2 j_{T0}^2}{\tau_T} \frac{(\mathrm{m/s^4})^2}{Hz} = \frac{\mathrm{m}^2}{\mathrm{s}^7} \tag{39}$$

The GLB of the rms miss for the ECJ target maneuver is derived from Eq. (15) and is written here as

$$Q = V_{\mathrm{go}}\begin{bmatrix} 2\sqrt{1+\frac{\sqrt{2}}{2}}\,\omega_o \kappa_{\mathrm{ECJ}}^2 & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}, \qquad \omega_o = \sqrt[8]{\frac{W_{jT}}{V_{\mathrm{go}}}}$$

$$\kappa_{\mathrm{ECJ}} = \kappa_{\mathrm{ECJ}}(\omega_o \tau_T) \tag{40}$$

where

$$\omega_o \tau_T = 1.75 \sqrt[8]{\frac{V_T^5}{D^2 T_s a_{T0}^3}} \tag{41}$$

In this case, the bound is

$$\text{rms miss}[m] \geq 0.57\, \kappa_{\mathrm{ECJ}}(\omega_o \tau_T) \sqrt[16]{\frac{T_s^7 D^{14} j_{T0}^2}{\tau_T}} = 0.56\, \kappa_{\mathrm{ECJ}}(\omega_o \tau_T) \sqrt[16]{\frac{T_s^7 D^{14} a_{T0}^5}{V_T^3}} \tag{42}$$

Figure 2 presents the correction factor $\kappa_{\mathrm{ECJ}}(\omega_o \tau_T)$ for the GLB miss distance computation for the ECJ maneuver versus $\omega_o \tau_T$.

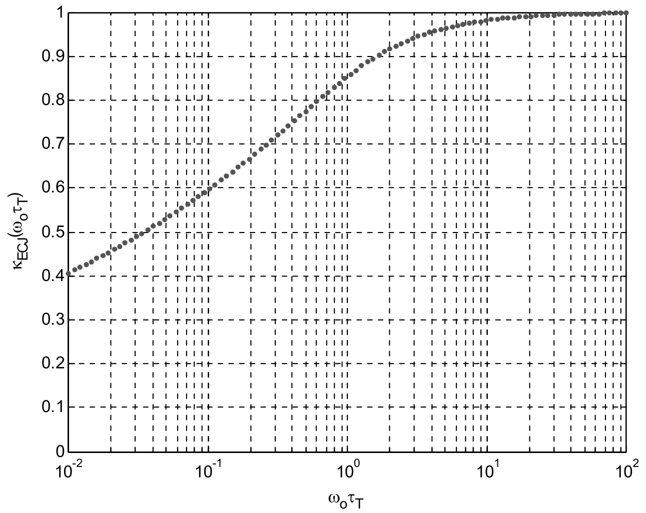

Fig. 2 The correction factor for the GLB miss distance computation for the ECJ maneuver.

X. Example

A. Step Target Acceleration

For step acceleration, the formula of the GLB can be written as

$$\text{GLB (rms miss)} = \sqrt{Q_{11}(t_f)} = \sqrt{2 V_{\mathrm{go}} \omega_o} = \sqrt{2} V_{\mathrm{go}}^{5/12}\left(\frac{a_{T0}^2}{t_m}\right)^{1/12} \tag{43}$$

for

$$V_{\mathrm{go}} = 1\ \frac{\mathrm{m}^2}{\mathrm{Hz}}, \qquad a_{T0} = 50\ \frac{\mathrm{m}^2}{\mathrm{s}}, \qquad t_m = 5\ \mathrm{s} \tag{44}$$

and

$$\text{GLB (rms miss)} = 2.37\ \mathrm{m} \tag{45}$$

This is exactly the value of the miss derived through simulation in [21]. This shows that the GLB in Eq. (15) is a tight bound.

B. Exponentially Correlated Acceleration Target

For the ECA maneuver, the following parameters are assumed:

$$T_s = 0.1\ \mathrm{s}, \qquad D = 10\ \mathrm{m} \tag{46}$$

$$a_{T0} = 50\ \mathrm{m/s^2}, \qquad V_T = 300\ \mathrm{m/s}$$

From Eq. (29) and Fig. 1,

$$\omega_o \tau_T = 11.17, \quad \text{so } \kappa_{\mathrm{ECA}} = 0.98 \tag{47}$$

and from Eq. (30), the GLB on the rms miss is 2.9 m.

XI. Conclusions

Explicit formulas of the GLB on the achievable miss of radar-guided missiles are presented. The GLB depends on a core set of parameters: the size, the acceleration level and velocity of the target, the period of the encounter; and the sampling rate. The formulas can be used for preliminary assessment of the missile performance.

Appendix A: Exponential Acceleration

This is a robust target-maneuver representation, as the following processes result in an exponential correlation function (behavior):

$$R(t) = R(0)\exp\left(-\frac{|t|}{\tau}\right) \tag{A1}$$

These processes are 1) the random-telegraph process [14,19], 2) the exponentially correlated stochastic process [17,18], and 3) the

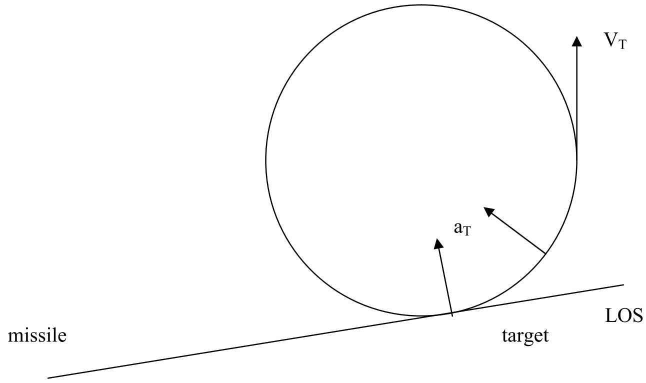

Fig. A1 Acceleration evolution description.

projection of a target acceleration onto the plane perpendicular to the target-missile LOS.

The following is a derivation of an estimate for the target time constant for case 3 above.

The projection of the targets' acceleration perpendicular to the LOS, as depicted in Fig. A1, is

$$a_T(t) = a_T(0)\cos[(\omega_T + \omega_{\mathrm{LOS}})t] \cong a_T(0)\cos(\omega_T t) \tag{A2}$$

To find the approximate time constant of this process, we set

$$a_T(\tau) = a_T(0)\cos(\omega_T \tau_T) \; = a_T(0)\exp(-1) \tag{A3}$$

thus,

$$\cos(\omega_T \tau_T) = \exp(-1) \tag{A4}$$

for which the solution is

$$\omega_T \tau_T = 1.2 \tag{A5}$$

We assume that the target performs a coordinated turn (circular path), then

$$\omega_T = \frac{a_T}{V_T} \tag{A6}$$

where a_T is the target acceleration, and V_T is the target velocity.

Thus, finally,

$$\tau_T = 1.2\frac{V_T}{a_T} \tag{A7}$$

Appendix B: Exponential Jerk

If we assume that the acceleration perpendicular to the LOS, as depicted in Fig. B1, is

$$\begin{aligned} a_T(t) &= a_T(0)\sin[(\omega_T + \omega_{\mathrm{LOS}})t] \cong a_T(0)\sin(\omega_T t) \\ j_T(t) &= \omega_T a_T(0)\cos(\omega_T t) \end{aligned} \tag{B1}$$

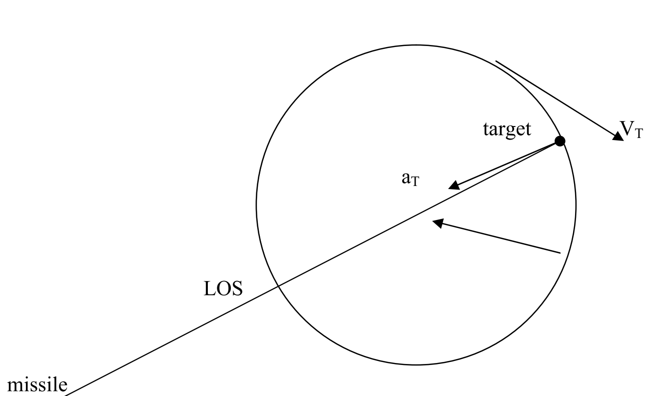

Fig. B1 Jerk evolution description.

To find the approximate time constant of this process,

$$j_T(\tau) = \omega_T a_T(0)\cos(\omega_T \tau_T) \; = \omega_T a_T(0)\exp(-1) \tag{B2}$$

Thus,

$$\cos(\omega_T \tau_T) = \exp(-1) \tag{B3}$$

for which the solution is Eq. (A4). Thus, for an ECJ,

$$j_{T0}(0) = \omega_T a_{T0}(0) \tag{B4}$$

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